Real Analysis II
Exercise Sheet 1: Topology I

Open, Closed, and Compact sets

Question 1. Decide whether the following sets are open or closed. Decide also which sets are compact. In each case provide a proof of the property or reasoning why the property fails.

- $(0, 1]$;
- $(0, \infty)$;
- $\{1, 2\}$;
- $\emptyset$;
- $\mathbb{R}$;
- $[-1, 2]$;
- $[-1, 2)$;
- $[0, 1] \cup \{2\}$;
- $[0, 1] \cup (2, 3]$;
- $(0, 1) \cup \{2\}$;
- $(0, 1) \cap \mathbb{Q}$;
- $\mathbb{Q} \cap [0, 1]$;
- $\mathbb{R} \setminus \mathbb{Q}$;
- $\mathbb{R} \setminus \{0\}$;
- $\{1/n \,|\, n \in \mathbb{N}\}$;
- $\{1/n \,|\, n \in \mathbb{N}\} \cup \{0\}$;
- $\{x \,|\, x^2 < 1\}$;
- $\{x \,|\, x^2 \leq 1\}$;
- $\{x \,|\, x > 1\}$;
- $\mathbb{Q} \cup \{\sqrt{2}\}$.

Limit points and connectedness

Question 2. For the following sets A decide whether x is a limit point:

- $A = (0, 1)$ and $x = 1$;
- $A = (0, 1)$ and $x = 1+\varepsilon$ for any $\varepsilon > 0$;
- $A = (0, 1) \cup \{2\}$ and $x = 2$;
- $A = (0, 1]$ and $x = 0$;
- $A = (0, 1]$ and $x = 1$;
- $A = [0, 1]$ and $x = 2$;
- $A = \mathbb{Z}$ and $x = 0$;
- $A = \mathbb{Z}$ and $x = 1/2$;
- $A = \{1/n \,|\, n \in \mathbb{N}\}$ and $x = 0$;
- $A = \{1/n \,|\, n \in \mathbb{N}\}$ and $x = 1$;
- $A = \{1/n \,|\, n \in \mathbb{N}\}$ and $x = 2$.

Question 3. For the following sets find the set of limit points

- $A = (0, 1) \cup \{2\}$;
- $A = (0, 1) \cup (3, 4)$;
- $A = (0, 1) \cup \{2, 3\}$;
- $A = \mathbb{Z}$;
- $A = \{1/n \,|\, n \in \mathbb{N}\}$;
- $A = \{0\} \cup \{1/n \,|\, n \in \mathbb{N}\}$;
- $A = \mathbb{Q} \cap (0, 1)$;
- $A = \mathbb{R} \setminus \mathbb{Q}$;
- $A = \big(\mathbb{Q} \cap [0, 1]\big) \cup \{2\}$;
- $A = [0, 1] \cap \big(\mathbb{R} \setminus \mathbb{Q}\big)$;
- $A = \{1, 1/2, 1/3, 1/4, \ldots\}$;
- $A = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$;
- $A = \{2 + 1/n \,|\, n \in \mathbb{N}\}$;
- $A = \{(-1)^n + 1/n \,|\, n \in \mathbb{N}\}$.

Question 4. Decide whether the following sets are connected or disconnected:

- $A = (0, 1)$;
- $A = [0, 1]$;
- $A = (0, 1) \cup (2, 3)$;
- $A = [0, 1] \cup \{2\}$;
- $A = [0, 1] \cup [1, 2]$;
- $A = [0, 1] \cup [2, 3]$;
- $A = \mathbb{Q} \cap (0, 1)$;
- $A = \mathbb{R} \setminus \mathbb{Q}$;
- $A = \mathbb{Z}$;
- $A = \{0\} \cup (1, 2)$;

Question 5. Decide whether the following statements are true or false.

- Every point of a set is a limit point of that set.
- A limit point of A must belong to A.
- A point of A need not be a limit point of A.
- A finite subset of $\mathbb{R}$ has no limit points.
- Every interval in $\mathbb{R}$ is connected.
- The union of two connected sets is connected.
- If A and B are separated then $A \cup B$ is disconnected.
- If $A \cap B = \emptyset$ then A and B are separated.
- If $A \cup B$ is disconnected then A and B are separated.
- $\mathbb{Q} \cap (0, 1)$ is connected.

Question 6. For the following combinations decide whether A and B are separated and whether $A \cup B$ is connected or disconnected:

- $A = [0, 1] \cup (2, 3)$ and $B = (1, 2)$;
- $A = \mathbb{Q} \cap (0, 1)$ and $B = [1/2, 1]$;
- $A = (0, 1)$ and $B = (1, 2)$;
- $A = (0, 1)$ and $B = [1, 2)$;
- $A = [0, 1]$ and $B = [1, 2]$;
- $A = (0, 1)$ and $B = (2, 3)$;
- $A = \mathbb{Q} \cap (0, 1)$ and $B = \big(\mathbb{R} \setminus \mathbb{Q}\big) \cap (0, 1)$;
- $A = (0, 1)$ and $B = \{1\}$;
- $A = [0, 1)$ and $B = [1, 2]$;
- $A = (0, 1)$ and $B = (1, 2)$;
- $A = (0, 1]$ and $B = (1, 2)$;
- $A = [0, 1]$ and $B = [1, 2]$;
- $A = [0, 1)$ and $B = (1, 2]$;
- $A = \mathbb{Q} \cap (0, 1)$ and $B = \{1\}$;
- $A = \mathbb{Q} \cap (0, 1)$ and $B = \big(\mathbb{R} \setminus \mathbb{Q}\big) \cap (0, 1)$;
- $A = (-\infty, 0)$ and $B = [0, \infty)$;
- $A = (-\infty, 0)$ and $B = (0, \infty)$.

Question 7. Consider the set

$$A = \{0\} \cup \{1/n \,|\, n \in \mathbb{N}\}.$$

- Which points of A are limit points of A?
- Find the set of all limit points of A.
- Is A closed?
- Is A compact?
- Is A connected?